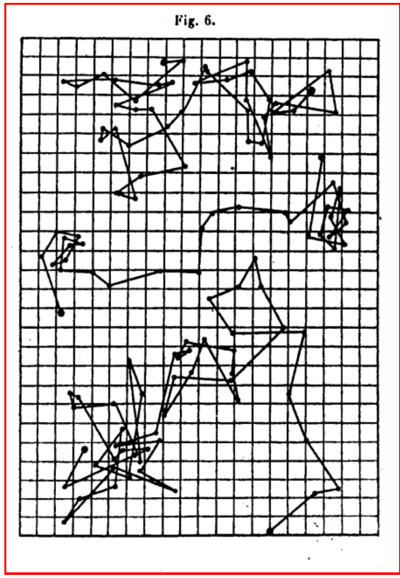
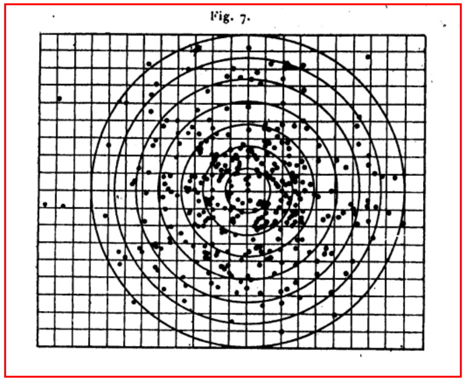


$$\frac{\partial}{\partial t} f(t,\vec{r}) = D\Delta f(t,\vec{r})$$

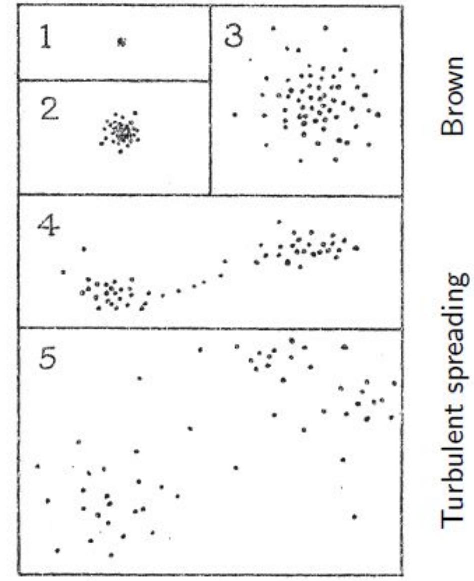


J.B. Perrin, 1909: **Brownian motion of small grains of putty**

$$\frac{\partial f}{\partial t} = D(t)\frac{\partial^2 f}{\partial l^2}$$

$$\boxed{D(t) \propto t^2}$$

Batchelor, 1952



Figs. 1–5.

Richardson, 1926: **dispersal of particles in the atmosphere vs dispersal of Brownian particles**

Brownian yet non-Gaussian Random Walk: from Superstatistics to Subordination of Diffusing Diffusivities

Aleksei Chechkin

Institute for Physics and Astronomy, University of Potsdam, and Akhiezer Institute for Theoretical Physics, National Science Center "Kharkov Istitute of Physics and Technology", Kharkov

Abram Fedorovich Ioffe
(1880-1960)
Hero of Socialist Labour

Ioffe Institute of Physics and Technology (1918)
26 Politekhnicheskaya, St Petersburg
194021, Russian Federation

Lev Davidovich Landau
(1908 – 1968)

National Scientific Center Kharkov Institute of Physics and Technology, Kharkov (1934)
Akademicheskaya st.1, 61108
Ukraine

Outline

Reminder

- **Brownian and Gaussian**
- **Anomalous, non-Brownian and non-Gaussian**
- **Variable diffusion processes**

Brownian yet non-Gaussian

→ **superstatistical Brownian motion**

→ **diffusing diffusivity model by Chubynsky and Slater (2014)**

→ **minimal diffusing diffusivity model (2017)**

- **Subordination concept**
- **Superstatistical behavior as a short time limit of subordination approach**
- **Solution of the bivariate Fokker-Planck equation via subordination approach**

A. Chechkin, F. Seno, R. Metzler, and I.M. Sokolov, PRX **7**, 021002 (2017)

Reminder. Brownian motion: massive particle in a heat bath (Langevin, 1908)

- **Langevin SDE**

$$\cancel{m\frac{d\mathrm{v}}{dt}} = -\frac{dU(x)}{dx} - \gamma m \mathrm{v} + \sqrt{2D_{\mathrm{v}}}\,\xi(t) \quad , \quad \frac{dx}{dt} = \mathrm{v}$$

$$\langle \xi(t)\xi(t')\rangle = \delta(t-t')$$

$\xi(t)$: Gaussian

Stokes' friction coefficient

$$\gamma = \frac{6\pi R}{m}\eta$$

$$D_{\mathrm{v}} = \gamma m k_B T$$

FDT

- **overdamped approximation**

$$\frac{dx}{dt} = -\frac{1}{m\gamma}\frac{dU}{dx} + \sqrt{2D_x}\,\xi(t)$$

$$D_x = \frac{k_B T}{m\gamma}$$

$D_x \to D, \quad m=1, \quad \gamma = 1$

- **Fokker-Planck eq**

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\frac{dU}{dx} f\right) + D\frac{\partial^2 f}{\partial x^2}$$

Reminder. Normal Brownian diffusion

No external force, U = 0

Fokker-Planck eq → **Diffusion eq**

- ∞ < x < + ∞

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

$$f(x,0) = \delta(x)$$

$$f(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)$$

Hallmarks of normal Brownian diffusion (Einstein, Smoluchowsky)

Mean squared displacement :

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} x^2 f(x,t)dx = 2Dt$$

Normal diffusion law

♦ **Wiener process: increments are (1) stationary, (2) Gaussian, (3) uncorrelated**

3 different sources of anomaly in the Langevin description $\left\langle \vec{r}^{\,2} \right\rangle \propto t^{\mu} \quad , \quad \mu \neq 1$

"Normal" :

$$\frac{dx}{dt} = -\frac{dU}{dx} + \sqrt{2D}\,\xi_G(t)$$

$\xi_G(t)$: Gaussian uncorrelated

$$\left\langle \xi_G(t)\xi_G(t') \right\rangle = \delta(t-t')$$

1 Lévy flights

$$\frac{dx}{dt} = -\frac{dU}{dx} + \sqrt{2D}\,\xi_L(t)$$

L ≡ Lévy

$\xi_L(t)$: non-Gaussian, uncorrelated

2 Fractional motions

$$\frac{dx}{dt} = -\frac{dU}{dx} + \sqrt{2D}\,\xi_H(t)$$

H ≡ Hurst

$\xi_H(t)$: Gaussian, strongly correlated

3 Variable diffusion processes

$$\frac{dx}{dt} = -\frac{dU}{dx} + \sqrt{2D(x,t)}\,\xi_G(t)$$

$\xi_G(t)$: Gaussian, uncorrelated

Random diffusivity

Motivation: **Brownian yet non-Gaussian diffusion** (S. Granick's group 2009, 2012)

A new class of diffusive dynamics has recently been reported:

- diffusion is normal
- PDF is non-Gaussian, typically characterized by a distinct exponential shape (Laplace distribution)

$$\left\langle \vec{r}^{\,2}(t) \right\rangle = \int_{-\infty}^{\infty} \vec{r}^{\,2} f(\vec{r},t)\, d\vec{r} = 2dDt \quad , \quad d = 1,2,3$$

$$P(\vec{r},t) \approx \exp\left(-\frac{|\vec{r}\,|}{\lambda(t)} \right) \quad , \quad \lambda = \sqrt{Dt}$$

- Peculiarities of Brownian diffusion in soft materials where the environment fluctuates slowly on broad timescales (Granick's group, PNAS2009, Nat Mat2012)
- Dynamics of tracer particles in colloidal hard-sphere suspensions (Kegel and Blaaderen, Science2000)
- Passive tracers in suspensions of eukaryotic swimmers, the alga Chlamydomonas reinhardtii (Goldstein et al., PRL2009)
- Confined diffusion of nanoparticles suspended in polymer solutions (Xue et al. JPCL 2016)
- The motion of individuals in heterogeneous populations, such as nematodes (~mm) in heterogeneous environments (agar) (Hapca et al., RSIF2009)

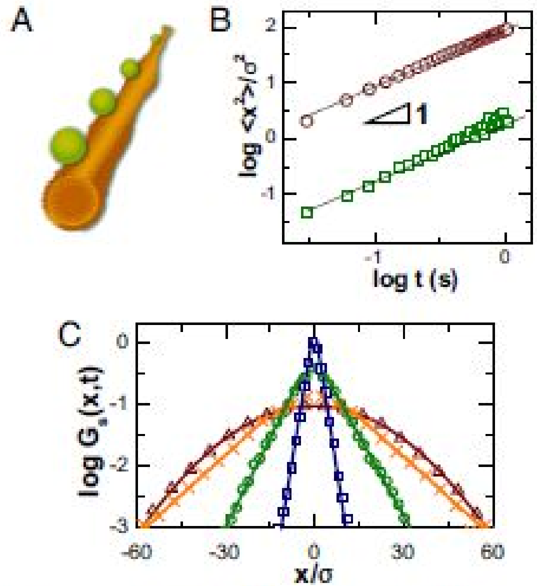
(For a more comprehensive list see Chechkin et al., PRX 2017)

Anomalous yet Brownian

Bo Wang[a], Stephen M. Anthony[b], Sung Chul Bae[a], and Steve Granick[a,b,c,d,1]

15160–15164 | PNAS | September 8, 2009 | vol. 106 | no. 36

Diffusion of colloids on phospholipid tubes



Exponential PDF at short times, Gaussian at longer times

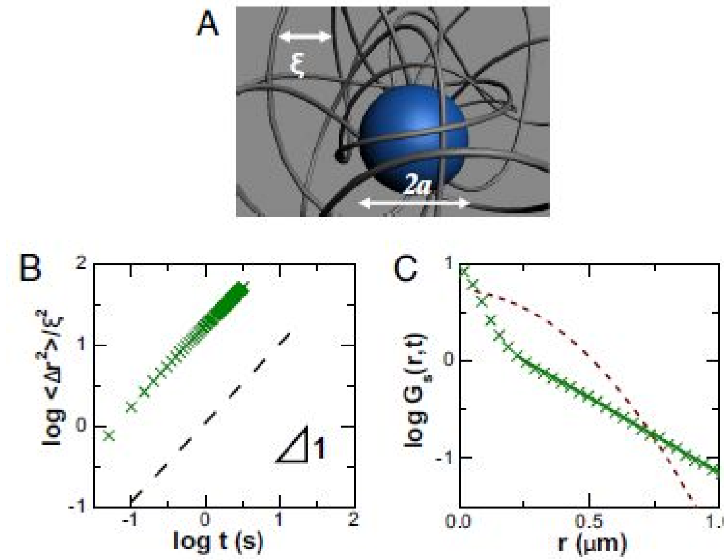
tubes. (A) Sche-
n, separated by
oid bilayers. (B)
r squared, plot-
omposed of pure
yers containing
ty. (C) From the
analysis of hundreds of trajectories without statistical difference, the displacement probability distribution of particles on lipid tubes composed of pure DLPC bilayers is plotted logarithmically against linear displacement normalized by particle diameter for several representative values of time step: 60 ms (squares), 0.6 s (circles), 3 s (crosses), and 5.8 s (triangles).

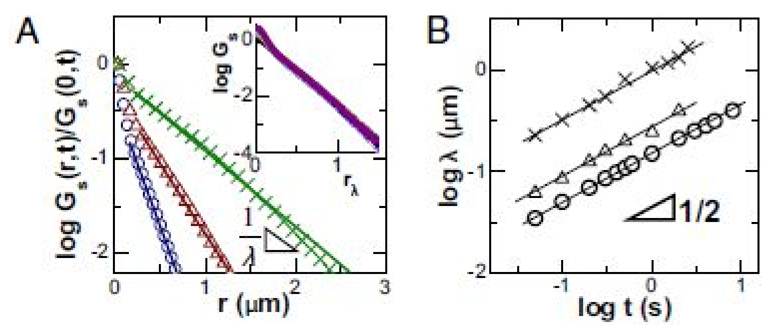
When Brownian diffusion is not Gaussian

Bo Wang, James Kuo, Sung Chul Bae and Steve Granick

NATURE MATERIALS | VOL 11 | JUNE 2012 | www.nature.com/naturematerials

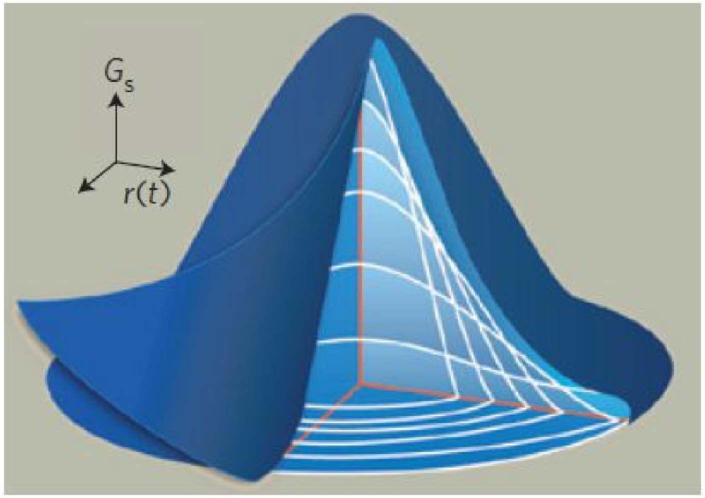
Colloidal beads in entangled actin suspensions





Granick et al.: Slow environmental relaxation is common in soft matter, as exemplified in experiments by colloidal particles diffusing in an environment of biopolymer filaments and phospholipid tube assemblies

b

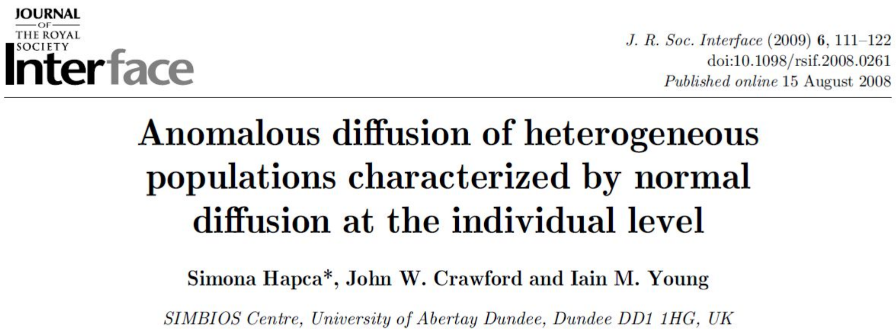


b, The distributions of displacements of objects diffusing a distance *r* in a certain time *t* in a slowly relaxing environment can be described with a non-Gaussian probability distribution function, $G_s(r,t)$, which can be decomposed into a set of elementary diffusive Gaussian processes (white curves).

c

c, Environmental fluctuations can span a wide range of times (or frequencies), as illustrated by the woodcut print *The Great Wave* from the Japanese artist Katsushika Hokusai.

(from Wang, Kuo, Bae & Granick, *Nat. Mat.* 2012)


JOURNAL OF THE ROYAL SOCIETY Interface

J. R. Soc. Interface (2009) 6, 111–122
doi:10.1098/rsif.2008.0261
Published online 15 August 2008

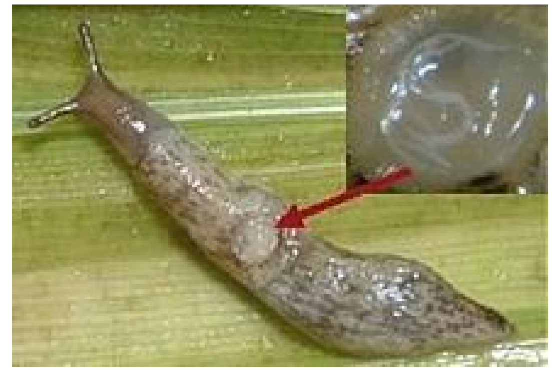
Anomalous diffusion of heterogeneous populations characterized by normal diffusion at the individual level

Simona Hapca*, John W. Crawford and Iain M. Young

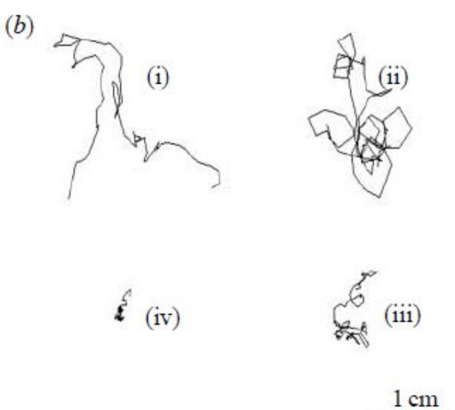
SIMBIOS Centre, University of Abertay Dundee, Dundee DD1 1HG, UK

10x magnification photo of Adult hermaphroditic female Parasitic nematode (*Phasmarhabditis hermaphrodita*). 1 Eye Piece Unit = 9.5µm

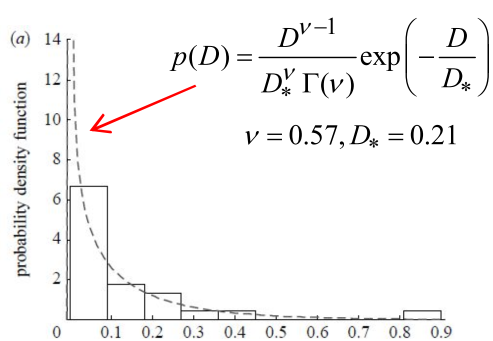
Slug pest *Deroceras reticulatum* is a common agricultural and horticultural pest and is one of the host species effected by *Phasmarhabditis hermaphrodita*



Deroceras reticulatum infected with slug parasitic roundworm *Phasmarhabditis hermaphrodita*.



Direct observations of nematode movement on agar plates suggested a large variation in individual behavior among the population



Diffusion is normal, but:

PDF of the diffusion coefficient

? How can this combination of normal, Brownian scaling of the MSD be reconciled with the existence of a non-Gaussian probability density function?

Superstatistical Brownian Motion

- Based on two statistical levels describing, respectively, the fast jiggly dynamics of the Brownian particle and the slow environmental fluctuations with spatially local patches of given diffusivity (Beck, 2001; Beck and Cohen, 2003; Beck, 2006)

- In math literature: compounding (Dubey, 1970)

$$f(x,t\,|\,D)=\frac{e^{-x^2/(4Dt)}}{\sqrt{4\pi Dt}} \;\Rightarrow\; f(x,t)=\int_0^\infty f(x,t\,|\,D)\,p(D)\,dD$$

Wang et al., 2009, 2012

Take $$p(D)=\frac{1}{D_*}\exp\left(-\frac{D}{D_*}\right) \Rightarrow f(x,t)=\frac{1}{2\sqrt{Dt}}\exp\left(-\frac{|x|}{\sqrt{Dt}}\right)$$

Hapca et al., 2009 : $$p(D)=\frac{D^{\nu-1}}{D_*^{\nu}\,\Gamma(\nu)}\exp\left(-\frac{D}{D_*}\right) \Rightarrow f(x,t)\sim\exp\left(-\frac{|x|}{\sqrt{Dt}}\right)$$

Similar phenomena for the case of anomalous diffusion

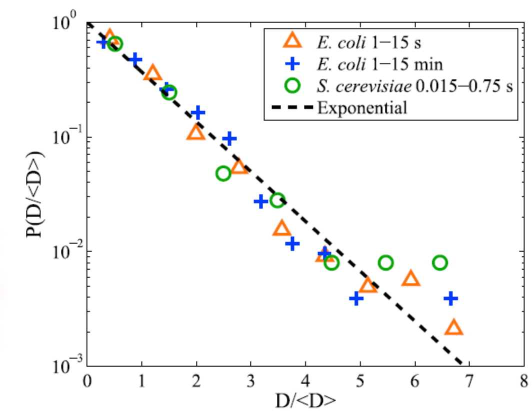
A.J. Spakowitz et al., 2017

RNA-protein particles in cellular cytoplasm exhibit subdiffusive behavior that is viscoelastic in its origin

$$\langle x^2(t)\rangle \simeq t^\alpha$$

$$G_\alpha(x,t|D_\alpha) = \frac{1}{\sqrt{4\pi D_\alpha t^\alpha}} \exp\left(-\frac{x^2}{4D_\alpha t^\alpha}\right)$$

FBM



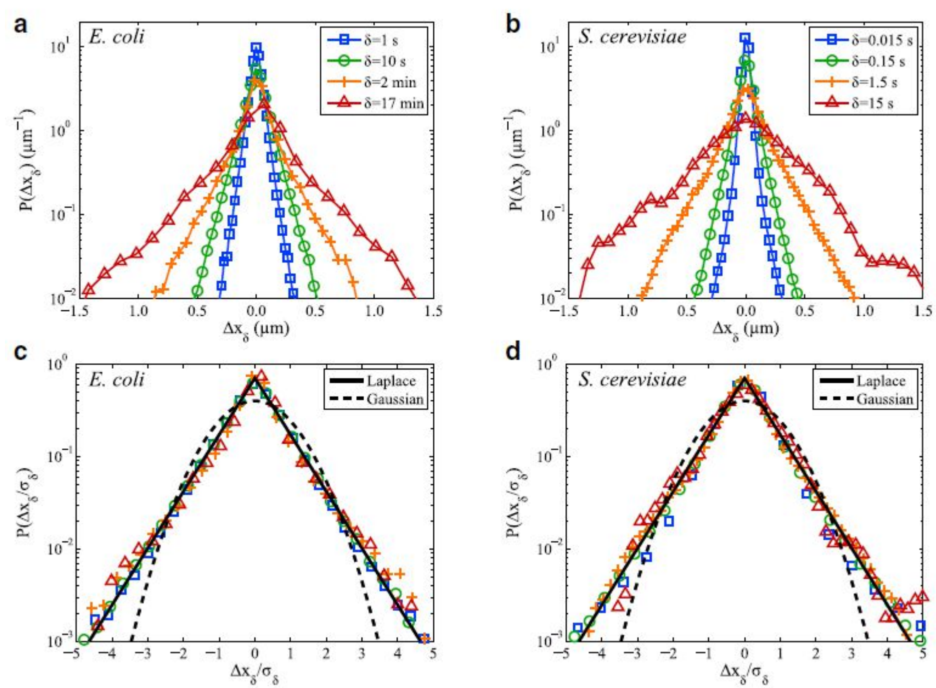
Diffusivities exhibit exponential distribution

$$p_D(D_\alpha) = \frac{1}{D_\alpha^\star} \exp\left(-\frac{D_\alpha}{D_\alpha^\star}\right)$$

Displacements exhibit a Laplace distribution

$$P(x,t) = \int_0^\infty p_D(D) G(x,t|D)\, dD$$

$$P_\alpha(x,t) = \frac{1}{\sqrt{4D_\alpha^\star t^\alpha}} \exp\left(-\frac{|x|}{\sqrt{D_\alpha^\star t^\alpha}}\right)$$



PHYSICAL REVIEW X 6, 021006 (2016)

Protein Crowding in Lipid Bilayers Gives Rise to Non-Gaussian Anomalous Lateral Diffusion of Phospholipids and Proteins

Jae-Hyung Jeon,[1,2] Matti Javanainen,[2,3] Hector Martinez-Seara,[2,4] Ralf Metzler,[2,5] and Ilpo Vattulainen[2,3,6]

We observe that **correlated Gaussian processes of the fractional Langevin equation type**, identified as the stochastic mechanism behind lipid motion in noncrowded bilayer, **no longer adequately describe the lipid and protein motion in crowded** but otherwise identical **membranes**. It turns out that protein crowding gives rise to a multifractal, non-Gaussian, and **spatiotemporally heterogeneous anomalous** lateral **diffusion** on time scales from nanoseconds to, at least, tens of microseconds. Our investigation strongly suggests that the macromolecular complexity and spatiotemporal membrane heterogeneity in cellular membranes play critical roles in determining the stochastic nature of the lateral diffusion and, consequently, the associated dynamic phenomena within membranes.

Superstatistical approach again (Ch, Seno, Metzler, Sokolov, 2016)

$$G_\alpha(x,t|D_\alpha) = \frac{1}{\sqrt{4\pi D_\alpha t^\alpha}} \exp\left(-\frac{x^2}{4D_\alpha t^\alpha}\right)$$

$$p_D(D_\alpha) = \frac{1}{\Gamma(1+1/\kappa)D_\alpha^\star} \exp\left(-\left[\frac{D_\alpha}{D_\alpha^\star}\right]^\kappa\right).$$

$$P(x,t) = \int_0^\infty p_D(D)G(x,t|D)dD$$

$$\Rightarrow \quad \boxed{P_\alpha(x,t) \simeq \exp\left(-c\left[\frac{|x|}{(4D_\alpha^\star t^\alpha)^{1/2}}\right]^{2\kappa/(1+\kappa)}\right)}$$

But: unable to describe a transition to Gaussian behavior at long times

The crossover from non-Gaussian to Gaussian can not be explained by superstatistical approach

$$P(x,t) = \int_0^\infty p_D(D) G(x,t|D) dD \quad (1) \qquad G(x,t|D) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right) \quad (2)$$

Fourier transforming (1): $P(k,t) = \int_0^\infty p_D(D) e^{-Dk^2 t} dD = \tilde{p}_D(s = k^2 t)$ Laplace transform of $p_D(D)$ at $s = k^2 t$ (3)

Inverse Fourier transform of (3) :

$$\begin{aligned} P(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\tilde{p}_D(k^2 t) dk \\ &= \frac{1}{2\pi t^{1/2}}\int_{-\infty}^{\infty} e^{-i\kappa x/t^{1/2}}\tilde{p}_D(\kappa^2) d\kappa \end{aligned} \quad (4)$$

The right hand side of (4) defines a scaling function *F* :

$$P(x,t) = \frac{1}{t^{1/2}} F(\zeta), \qquad \zeta = x/t^{1/2}. \quad (5)$$

The form *F* as function of the similarity variable ζ is an invariant

Important step forward:

Chubynsky-Slater bi-Langevin model (2014): diffusing diffusivity

the very idea was first introduced in financial math (“stochastic volatility”)

$$\boxed{\frac{dx}{dt} = \sqrt{2D(t)}\ \xi(t)}$$

$$\boxed{\frac{dD}{dt} = -g + \sigma\,\eta(t)}$$

$D(t)$: Brownian particle in a gravitational field

$$\frac{\partial}{\partial t} p(D,t) = g\frac{\partial}{\partial D} p(D,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial D^2} p(D,t) \qquad \text{FPE}$$

Reflecting boundary at $D = 0$

$$J(D,t=0)=0 \qquad J(D,t) = -g\,p(D,t) - \frac{\sigma^2}{2}\frac{\partial}{\partial D} p(D,t)$$

Stationary solution of the FPE (barometric formula) :

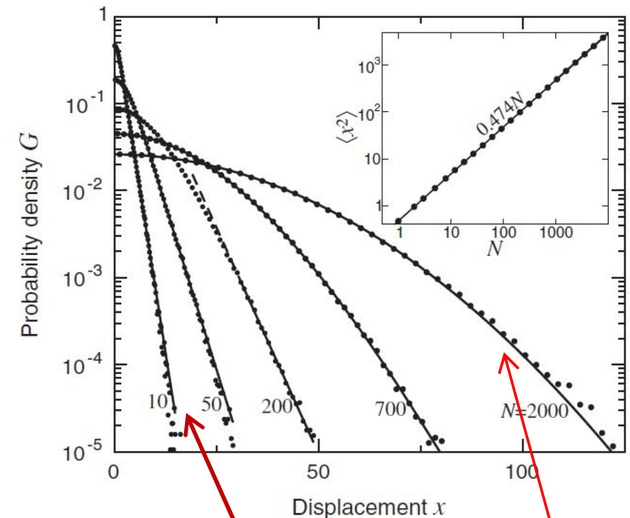
$$\boxed{p_{st}(D) = \frac{1}{D_*}\exp\left(-\frac{D}{D_*}\right)} \qquad \boxed{D_* = \frac{\sigma^2}{2g}}$$

Then, the superstatistical approach:

$$p(x,t) = \int_0^\infty dD\ p_{st}(D)\frac{e^{-x^2/(4Dt)}}{\sqrt{4\pi Dt}} = \frac{1}{2\sqrt{D_* t}}\exp\left(-\frac{|x|}{\sqrt{D_* t}}\right)$$

$$\boxed{\left\langle x^2(t)\right\rangle = \frac{\sqrt{\pi}}{4}D_* t}$$

Langevin simulations



Exponential at short times

Gaussian at long times

BUT: Langevin simulations give exponential PDF at short times (covered by superstatistical approach) and Gaussian PDF at long times **TRANSITION TIME ???**

Minimal Langevin model for diffusing diffusivities (Chechkin et al. 2017)

$$\boxed{\frac{d\vec{\boldsymbol{r}}}{dt} = \sqrt{2D(t)}\vec{\boldsymbol{\xi}}(t)} \qquad \left\langle \vec{\boldsymbol{\xi}}(t) \right\rangle = \left\langle \vec{\boldsymbol{\eta}}(t) \right\rangle = 0 \quad , \quad \left\langle \xi_i(t_1)\xi_j(t_2) \right\rangle = \delta_{ij}\delta\left(t_1 - t_2\right), \quad i,j = x,y,z$$

$$\boxed{D(t) = \vec{\boldsymbol{Y}}^2(t)} \qquad \left\langle \eta_l(t_1)\eta_m(t_2) \right\rangle = \delta_{l\,m}\delta\left(t_1 - t_2\right) \quad l,m = 1,2,\dots,n$$

$$x(0) = 0$$

$$y(0) = y_0$$

$$\boxed{\frac{d\vec{\boldsymbol{Y}}}{dt} = -\frac{\vec{\boldsymbol{Y}}}{\tau} + \sigma\vec{\boldsymbol{\eta}}(t)}$$

Y(t): n-dim. Ornstein – Uhlenbeck process

Reasonings:

1. On the single trajectory level the particle motion is modeled with a random diffusivity D(t).

2. Taking D(t) as the square of the auxiliary variable Y(t) guarantees the non-negativity of D(t) .

Why OU ?

3. Makes sure that the diffusivity dynamics is stationary with a given correlation time τ.

4. Ensuing PDF $p_D(D)$ has exponential tails thus guaranteeing the emergence of the Laplace-like PDF P(r,t) at short times.

5. The above set of equations also allows for an analytical solution !

6. The number of modes *n* is essentially a free parameter of the model – additional flexibility

Minimal Langevin model in dimensionless units $t \to t/\tau \quad \vec{r} \to \vec{r}/(\sigma\tau)$

$$\boxed{\frac{d\vec{r}}{dt} = \sqrt{2D(t)}\vec{\xi}(t)} \qquad \langle\vec{\xi}(t)\rangle = \langle\vec{\eta}(t)\rangle = 0 \quad , \quad \langle\xi_i(t_1)\xi_j(t_2)\rangle = \delta_{ij}\delta(t_1 - t_2), \quad i,j = x,y,z$$

$$\boxed{D(t) = \vec{Y}^2(t)} \qquad \langle\eta_l(t_1)\eta_m(t_2)\rangle = \delta_{lm}\delta(t_1 - t_2) \quad l,m = 1,2,\ldots,n$$

$$x(0) = 0$$

$$y(0) = y_0$$

$$\boxed{\frac{d\vec{Y}}{dt} = -\vec{Y} + \vec{\eta}(t)}$$

y(t): n-dim. Ornstein – Uhlenbeck process

Solution of OU : $$\mathbf{Y}(t) = \mathbf{Y}_0 e^{-t} + \int_0^t \eta(t')e^{-(t-t')}dt'. \qquad \langle\mathbf{Y}(t_1)\mathbf{Y}(t_2)\rangle \sim \frac{n}{2}e^{-|t_2-t_1|}$$

FPE for OU : $$\frac{\partial}{\partial t}f(\mathbf{Y},t) = \frac{\partial}{\partial \mathbf{Y}}\Big(\mathbf{Y}f(\mathbf{Y},t)\Big) + \frac{1}{2}\frac{\partial^2}{\partial \mathbf{Y}^2}f(\mathbf{Y},t). \qquad f_{st}(\mathbf{Y}) = \frac{1}{\pi^{n/2}}e^{-\mathbf{Y}^2}.$$

Stationary diffusivity distribution

$$p_D^{st}(D) = \int_{-\infty}^{\infty} f_{st}(Y)\delta\left(D - Y^2\right)dY$$

$$p_D^{st}(D) = \begin{cases} \dfrac{e^{-D}}{\sqrt{\pi D}} \quad , & n = 1 \\ e^{-D} \quad , & n = 2 \\ \dfrac{2\sqrt{D}}{\sqrt{\pi}}e^{-D} \quad , & n = 3 \end{cases}$$

Exponential dependence is dominating

Digression. Subordination concept in probability theory

(Bochner, 1949, Feller, vol.II)

- Beautiful mathematical theory widely used e.g., in financial mathematics

"Random time change"

$x(\tau)$ Parent process x with PDF G(x,τ) evolving in operational ***random*** time τ

$\tau(t)$ Directing process, or subordinator, with PDF $T(\tau,t)$: operational time evolving in physical time t *(also: path variable τ)*

$x(\tau(t))$ Subordinated process controlled by subordinator $\tau(t)$

Integral formula for subordination (Feller, vol.II) ⇨

$$\boxed{p(x,t)=\int_0^\infty d\tau\, G(x,\tau)T(\tau,t)}$$

The probability for a walker to arrive at position x at time t equals the probability of being at τ on the path at time t, multiplied by the probability of being at position x for this path length τ, summed over all path lengths H. Fogedby (1994)

CTRW: random number of steps $n(t) \leftrightarrow$

$\leftrightarrow$ operational time $\tau(t)$

Solution of the minimal model in the form of subordination integral

$$\boxed{\frac{d\vec{r}}{dt} = \sqrt{2D(t)}\vec{\xi}(t)} \quad \text{(a)}$$

$$\boxed{D(t) = \vec{Y}^2(t)} \quad \text{(b)}$$

$$\boxed{\frac{d\vec{Y}}{dt} = -\vec{Y} + \vec{\eta}(t)} \quad \text{(c)}$$

$$\frac{\partial}{\partial t}\overline{P}(\mathbf{r},t) = D(t)\nabla^2\overline{P}(\mathbf{r},t).$$

$$\tau(t) = \int_0^t dt' D(t') = \int_0^t dt' Y^2(t')$$

Langevin eq.(a) in the subordinated form

$$\left[\ \boxed{\frac{d\vec{r}}{d\tau} = \sqrt{2}\vec{\xi}(\tau)} \quad \boxed{\frac{d\tau}{dt} = D(t)}\ \right]$$

$\boxed{\vec{r}(\tau)}$ Parent process ***r*** evolving in operational ***random*** time τ

$$G(\vec{r},\tau) = \frac{1}{\sqrt{4\pi\tau}}\exp\left(-\frac{r^2}{4\tau}\right)$$

$\tau(t)$ Directing process, or subordinator, with PDF $T(\tau,t)$:
operational time evolving in physical time t *(also: path variable* τ *)*

Integral formula for subordination :

$$\boxed{P(\vec{r},t) = \int_0^\infty d\tau\, G(\vec{r},\tau) T_n(\tau,t)}$$

!

BUT:
What is
$T_n(\tau,t)$???

Distribution of the integrated square of the Ornstein-Uhlenbeck process (T. Dankel, SIAM J. Appl. Math.1991).

Laplace transform of $T_1(\tau,t)$ $$\tilde{T}_1(s,t) = \frac{e^{t/2}}{\left[\frac{1}{2}\left(\sqrt{1+2s}+\frac{1}{\sqrt{1+2s}}\right)\sinh\left(t\sqrt{1+2s}\right)+\cosh\left(t\sqrt{1+2s}\right)\right]^{1/2}}$$

Integral formula for subordination again:

$$\boxed{P(\vec{r},t) = \int_0^\infty d\tau\, G(\vec{r},\tau) T_n(\tau,t)}$$

Langevin eq.(a) in the subordinated form

Fourier transform of P via Laplace transform of T :

$$\boxed{P(\vec{k},t) = \int_0^\infty d\tau\, G(\vec{k},\tau) T_n(\tau,t) = \int_0^\infty d\tau\, e^{-k^2\tau} T_n(\tau,t) = \tilde{T}_n(s=k^2,t)}$$

For n-dimensional OU $D(t) = \vec{Y}^2(t)$ $\tau(t) = \int_0^t \mathrm{Y}^2(t)dt' = \int_0^t (Y_1^2(t')+Y_2^2(t')+\ldots+Y_n^2(t'))\,dt'$

$$\tilde{T}_n(s,t) = \exp(nt/2)\Big/\left[\frac{1}{2}\left(\sqrt{1+2s}+\frac{1}{\sqrt{1+2s}}\right)\right.$$

$$\left.\times\sinh\left(t\sqrt{1+2s}\right)+\cosh\left(t\sqrt{1+2s}\right)\right]^{n/2}.$$

Brownian mean squared displacement and leptokurtic behavior

(isotropic case)

$$\langle \mathbf{r}^2(t)\rangle = -\nabla^2_{\mathbf{k}}\hat{P}(\mathbf{k},t)\Big|_{\mathbf{k}=0},$$
$$\langle \mathbf{r}^4(t)\rangle = \nabla^4_{\mathbf{k}}\hat{P}(\mathbf{k},t)\Big|_{\mathbf{k}=0}.$$

$$\nabla^2_{\mathbf{k}} = \frac{1}{k^{d-1}}\frac{\partial}{\partial k}\left(k^{d-1}\frac{\partial}{\partial k}\right).$$

$$\hat{P}(\mathbf{k},t) = \int_0^\infty e^{-k^2\tau}T_n(\tau,t)d\tau = 1 - k^2\int_0^\infty \tau T_n(\tau,t)d\tau + \frac{k^4}{2}\int_0^\infty \tau^2 T_n(\tau,t)d\tau + \dots$$

$$\langle \mathbf{r}^2(t)\rangle = 2d\int_0^\infty \tau T_n(\tau,t)d\tau = 2d\langle\tau\rangle = -2d\left.\frac{\partial \tilde{T}_n(s,t)}{\partial s}\right|_{s=0}$$

$$\langle \mathbf{r}^4(t)\rangle = 4d(2+d)\int_0^\infty \tau^2 T_n(\tau,t)d\tau = 4d(2+d)\langle\tau^2\rangle = 4d(d+2)\left.\frac{\partial^2 \tilde{T}_n(s,t)}{\partial s^2}\right|_{s=0}$$

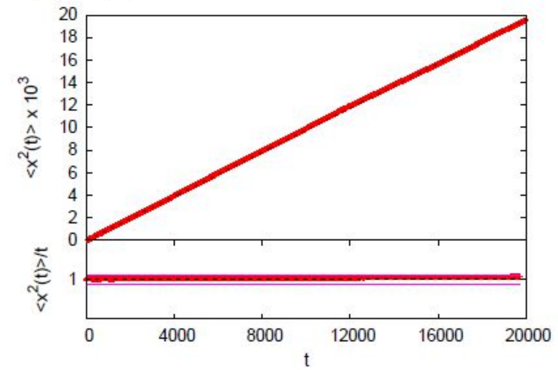
$$\langle \mathbf{r}^2(t)\rangle = dnt = 2d\langle D\rangle_{\mathrm{st}}t,$$ where $$\langle \mathbf{Y}^2(t)\rangle = \langle D\rangle_{\mathrm{st}} = \frac{n}{2}.$$

$$\langle \mathbf{r}^4(t)\rangle = 4d(2+d)\langle D\rangle_{\mathrm{st}}\left[-\frac{1-e^{-2t}}{2} + t + \langle D\rangle_{\mathrm{st}}t^2\right]$$

Brownian MSD and leptokurtic behavior - 2

In dimensional units $\langle \mathbf{r}^2(t)\rangle = dn\sigma^2\tau t = 2d\langle D\rangle_{\rm st}D_\star t.$ linear in *t* for **all** times

parameters of OU processes for diff diff



Kurtosis $$K = \frac{\langle \mathbf{r}^4(t)\rangle}{\langle \mathbf{r}^2(t)\rangle^2}.$$

Short times t < τ : n = d

$$K \sim \left(1+\frac{2}{d}\right)\left(1+\frac{1}{\langle D\rangle_{\rm st}}\right) = \begin{cases} 9, & d=1 \\ 4, & d=2 \\ 25/9, & d=3 \end{cases}$$

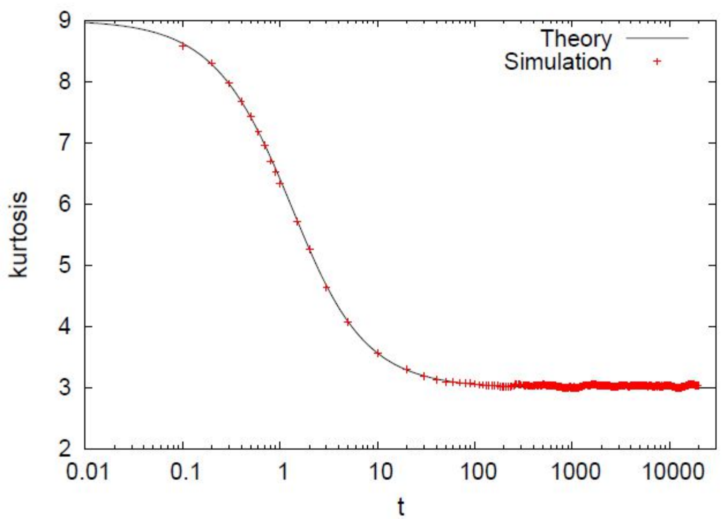
Long times t > τ: Gaussian value

does not depend on n

$$K \sim \left(1+\frac{2}{d}\right) = \begin{cases} 3, & d=1 \\ 2, & d=2 \\ 5/3, & d=3 \end{cases}$$

$$K \sim 1+\frac{2}{d}+\frac{2+d}{d\langle D\rangle_{\rm st}t}.$$

$\propto$ 1/t



♦ Solution of the Minimal Model at short times, $t \ll 1$, d = n

$t \ll \tau$

d = 1

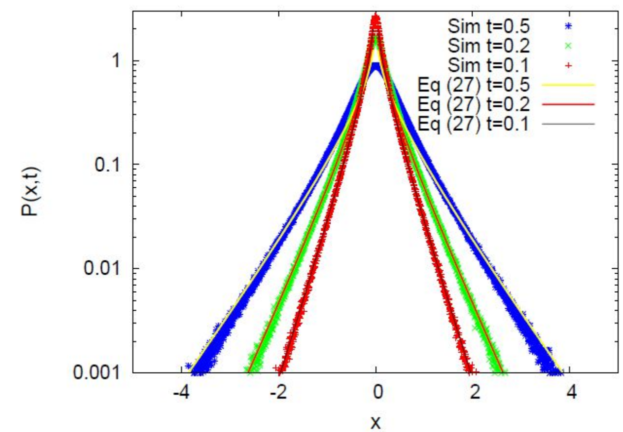
$$P(x,t) \simeq \frac{1}{\pi\sqrt{t}} \int_0^\infty dk \frac{\cos(kx)}{\left(k^2 + \frac{1}{t}\right)^{1/2}} = \frac{1}{\pi\sqrt{t}} K_0\left(\frac{x}{\sqrt{t}}\right) \Rightarrow P(x,t) \sim \frac{1}{\sqrt{2\pi |x| \sqrt{t}}} \exp\left(-\frac{|x|}{\sqrt{t}}\right)$$

d = 2

$$P(\mathbf{r},t) = \frac{1}{2\pi t} K_0\left(\frac{r}{\sqrt{t}}\right).$$

d = 3

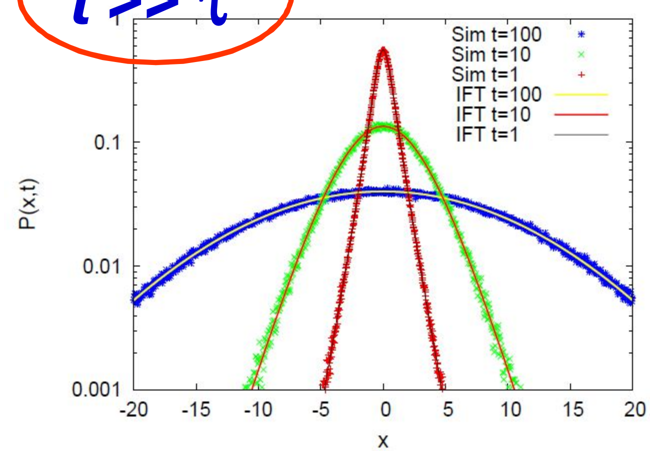
$$P(\mathbf{r},t) = \frac{1}{2\pi^2 t^{3/2}} K_0\left(\frac{r}{\sqrt{t}}\right)$$



♦ Solution of the Minimal Model at long times $t \gg 1$, d = n

$t \gg \tau$

$$P(\mathbf{r},t) \sim \frac{1}{(4\pi\langle D\rangle_{\mathrm{st}} t)^{n/2}} \exp\left(-\frac{\mathbf{r}^2}{4\langle D\rangle_{\mathrm{st}} t}\right).$$



Exponential at short times ⇒ **Gaussian at long times**

♦ Relation to superstatistical Brownian motion

$$P_s(\mathbf{r},t) = \int_0^\infty p_D^{\mathrm{st}}(D) G(\mathbf{r},t|D) dD.$$

$$D(t) = \vec{Y}^2(t)$$

$$D_\star = \sigma^2 \tau$$

d = 1

$$P_s(x,t) = \frac{1}{2\pi\sqrt{D_\star t}} \int_0^\infty \frac{1}{D} \exp\left(-\frac{D}{D_\star} - \frac{x^2}{4Dt}\right) dD = \frac{1}{\pi\sqrt{D_\star t}} K_0\left(\frac{|x|}{\sqrt{D_\star t}}\right)$$

d = 2

$$P_s(\mathbf{r},t) = \frac{1}{4\pi D_\star t} \int_0^\infty \frac{1}{D} \exp\left(-\frac{D}{D_\star} - \frac{r^2}{4Dt}\right) dD = \frac{1}{2\pi D_\star t} K_0\left(\frac{r}{\sqrt{D_\star t}}\right)$$

d = 3

$$P_s(\mathbf{r},t) = \frac{2}{\pi^2 (4D_\star t)^{3/2}} \int_0^\infty \frac{1}{D} \exp\left(-\frac{D}{D_\star} - \frac{r^2}{4Dt}\right) dD = \frac{1}{2\pi^2 (D_\star t)^{3/2}} K_0\left(\frac{r}{\sqrt{D_\star t}}\right)$$

→ **Superstatistics is valid at times less than the correlation time of the random diffusivity**

Perfect coincidence :
superstatistics ≡ subordination for any d and n at short times!

The short time regime of the subordination formalism leads directly to the superstatistical result.

- $t < \tau$: diffusion coefficient does not change considerably, and the subordination scheme describes an ensemble of particles, each diffusing with its own diffusion coefficient.

- This mimics a spatially inhomogeneous situation, when the local diffusion coefficient is random, but stays constant within confined spatial domain $\Rightarrow$ ensemble of particles moving in different domains exhibits a superstatistical behavior .

- $t > \tau$: the particle goes away from the local patch of diffusivity D thus violating the assumption of the superstatistical approach .

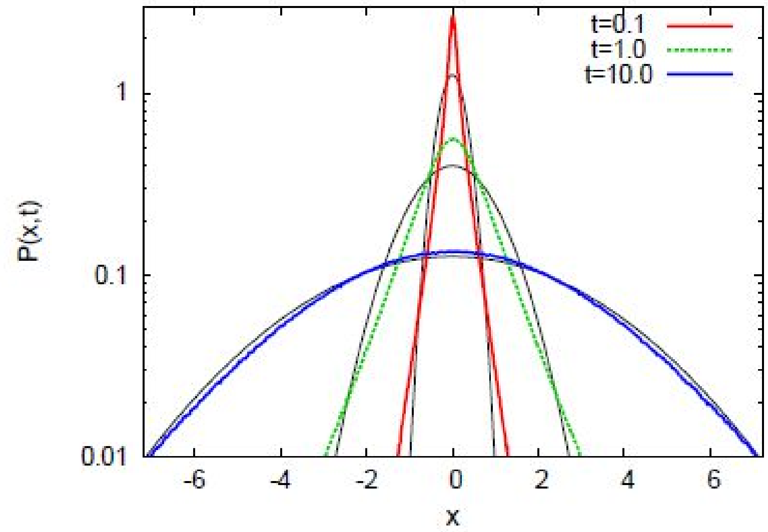


FIG. 5: Probability density function $P(x,t)$ for $d = n = 1$ from simulations of the Langevin equations (20) for three different times. Comparison with Gaussian distribution (79) demonstrates the strongly non-Gaussian behavior at short times and the almost fully Gaussian shape at longer times.

- The subordination approach delivers an approximation to a spatially disordered situation and adequately describes the transition to a Gaussian behavior.

♦ Fokker-Planck equation and relation to subordination approach

$$\frac{dx}{dt} = \sqrt{2D(t)}\,\xi(t)$$

$$D(t) = Y^2(t)$$

$$\frac{dY}{dt} = -Y + \eta(t)$$

Bivariate FPE :

$$\frac{\partial}{\partial t} f(x,Y,t) = \mathcal{L}_Y f(x,Y,t) + Y^2 \frac{\partial^2}{\partial x^2} f(x,Y,t) \qquad \mathcal{L}_Y = \frac{\partial}{\partial Y} Y + \frac{1}{2}\frac{\partial^2}{\partial Y^2}$$

Marginal PDF : $p(x,t) = \int_{-\infty}^{\infty} dy\, f(x, Y$

Ansatz: $\boxed{f(x,Y,t) = \int_0^{\infty} d\tau G(x,\tau) q(\tau,Y,t)}$ where PDF $q(\tau$,Y,$t)$ $\boxed{\frac{dY}{dt} = -Y + \eta(t)}$ $\boxed{\frac{d\tau}{dt} = Y^2}$

FPE for $q(\tau$,Y,$t)$ $\boxed{\frac{\partial}{\partial t} q(x,Y,t) = \mathcal{L}_Y q(x,Y,t) - Y^2 \frac{\partial}{\partial \tau} q(x,Y,t)}$

Marginal PDF : $\boxed{p(x,t) = \int_0^{\infty} d\tau\, T(\tau,t) G(x,\tau)}$ where $\boxed{T(\tau,t) = \int_{-\infty}^{\infty} dY\, q(\tau,Y,t)}$

Verification: $\boxed{\frac{\partial f(x,Y,t)}{\partial t} = \int_0^{\infty} d\tau\, G(x,\tau) \frac{\partial}{\partial t} q(\tau,Y,t) = \mathcal{L}_Y f(x,Y,t) + Y^2 \frac{\partial^2}{\partial x^2} f(x,Y,t)}$ OK

Σ

- ♥ **The subordination formulation of the diffusing diffusivity problem – 1st main result**

- ♥ **The full consistency in the short time limit between the subordination approach and superstatistics – 2nd main result**

- ♥ **The explicit derivation for the crossover to a Gaussian behavior – 3rd main result**

- ♥ **The connection of the bivariate Fokker-Planck equation for the double Langevin system with the subordination approach – 4th main result**

- ♦ The model can be calibrated with respect to two parameters, diffusivity correlation time τ and the amplitude σ, thus making this model useful for experimentalists

- ♦ Subordination: a promising tool to go beyond the Brownian yet non-Gaussian process and consider anomalous diffusion

From a practical viewpoint:

Why all this matters ? As the non-Gaussian character mostly appears in the tails of the distribution, and thus with low probability…

The answer is: this can be at the heart of the understanding of soft-matter systems, especially in those where the rare events dominate the long-time dynamics

- Relaxation, transport and reaction in complex media
- Triggered actions in which diffusion, in a first step of a process, sets off a cascade of subsequent process, such as signalling in biology or kinetically controlled processes
- The emerging field of active matter with internally driven systems that sample rare events more frequently
- Systems with first passage processes – such as targeting, translocation, triggering, criticality,… - rare fluctuations may dominate the system's dynamics and induce relevant transitions

Certainly, Fickian yet non-Gaussian diffusion will lead us

to the discovery of unexpected phenomena

Many thanks to my collaborators:

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Different approaches, but similar results in
- Jain and Sebastian (2017)
-Tyagi and Cherail (2017)
- Y. Lanoiselée and D. Grebenkov (2017)

Thank you for attention !